# Student seminar exercise sheet Week 1

1. Let $G$ be a finite group of automorphisms of a ring $A$ and let

$$A^G = \{a \in A \mid \sigma(a) = a \text{ for all } \sigma \in G\},$$

be the ring of invariants.

   (a) Show that $A$ is integral over $A^G$.

   (b) Let $\mathfrak{p} \subset A^G$ be a prime ideal and $P$ the set of prime ideals $\mathfrak{q} \subset A$ such that $\mathfrak{q} \cap A^G = \mathfrak{p}$. Show that $G$ acts transitively on $P$.
   *Hint: See exercises 5.12 and 5.13 in Atiyah-Macdonald's Introduction to commutative algebra.*

2. Show that the ideal $J = (2, 1 + \sqrt{-5})$ is not principal in $\mathbb{Z}[\sqrt{-5}]$. In particular the ideal class group of $\mathbb{Q}[\sqrt{-5}]$ is non-trivial.

3. Give examples of extensions of number fields $K/F$ and prime ideals $\mathfrak{p} \subset \mathcal{O}_F$ that are unramified, ramified, inert and completely split in $K/F$.

4. Show that the cyclotomic polynomial $\Phi_n \in \mathbb{Z}[X]$ factors into distinct linear factors modulo a prime $p$ if any only if $p \equiv 1 \pmod{n}$.

   *Hint: $\mathbb{F}_p^\times$ contains a cyclic group of order $n$ if and only if $p \equiv 1$ (mod $n$).*